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Spectra of the Orr–Sommerfeld equation: the plane Poiseuille flow for the normal fluid revisited

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Abstract

We further investigate the strange spectra of the Orr–Sommerfeld operator using the plane Poiseuille flow as a basic stationary flow for normal fluids in the two-fluid system of helium II by a verified preconditioned complex-matrix solver. The strange spectra are composed of one pair of eigenvalues with the same phase speed (real part) but different amplification factors (imaginary part) corresponding to the specific Reynolds number and wavenumber we select. These kinds of degeneracy disappear for Reynolds number around 400, where the 'drifting' of the complete spectra imposes much more complexity on the searching.

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The recent papers of Baggett *et al* [1], Jackson *et al* [2] and Elofsson and Alfredsson [3] revived an interest in the study of the detailed spectra of the linear stability equation (Orr–Sommerfeld (OS) equation; the basic flow could be shear flow or plane Poiseuille flow) for the description of the hydrodynamical transition to turbulence in normal fluids (instead of superfluids). Plane Poiseuille flow is one of the fundamental base-flow types for the wall-bounded parallel-flow-instability research regime. The usual approach to considering linear stability is through the OS equation. Following the usual assumptions of linearized stability theory, we have $v_i(x_i, t) = \bar{v}_i(x_i) + v_i'(x_i, t)$, and similarly, $p(x_i, t) = \bar{p}(x_i) + p'(x_i, t)$ for the velocity and pressure terms in the incompressible Navier–Stokes equations. Then by substituting these into the dimensionless two-dimensional Navier–Stokes equation, and eliminating the pressure terms, the linearized equation or so-called OS equation, which governs the variation of the disturbances, is

$$(D^2 - \alpha^2)^2\phi = \mathrm{i}\alpha R[(\bar{u} - c)(D^2 - \alpha^2)\phi - (D^2\bar{u})\phi] \tag{1}$$

where $R = \rho u_{\max} h/\mu$ is the Reynolds number based on the half channel-width and $\bar{u} = 1 - y^2$ is the (mean) basic velocity profile of the flow. The stream function for the disturbance, Ψ, such that $u' = -\partial\Psi/\partial y$, $v' = -\partial\Psi/\partial x$, may be assumed to have the form $\Psi(x, y, t) = \phi(y)\exp[\mathrm{i}\alpha(x - ct)]$ in the usual normal-mode analysis, α is the wavenumber (real) and c is $c_\mathrm{r} + \mathrm{i}c_\mathrm{i}$. This is a kind of Tollmien–Schlichting transversal wave: c_r is the ratio between the velocity of propagation of the wave of perturbation and the characteristic velocity, c_i is called

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the amplification factor and α is equal to $2\pi L^{-1}$, where L is the wavelength of the Tollmien–Schlichting perturbation [4]. Boundary conditions are $\phi(1) = \phi'(1) = \phi(-1) = \phi'(-1) = 0$. In the usual temporal stability problem, in which the growth or decay of a disturbance in time is considered, we take α and R to be real and then treat the (complex) wave-speed c as the eigenvalue parameter of the problem.

However, the OS operator is non-normal [5–7], so the eigenfunctions, though complete, may be nearly linearly dependent and the eigenvalues may be highly sensitive to perturbations [8] (even though our previous attempt [8], in parts, had matched the observation that the propagation speed of the front part of a certain turbulent spot is the same as two-thirds of the centre-line velocity in plane Poiseuille flow). These would induce many difficulties if we want to start with numerical approaches, such as spectral methods, even though this method is well known to be essentially very accurate [9] for certain cases. For example, a previous linear stability approach by Orszag in 1971 [10] showed that plane Poiseuille flow is stable if the Reynolds number is less than the critical one $R_c \sim 5772$.

However, recent research, inspired by the works in [1–3], did show that R_c could be much smaller than 5772 once other mechanisms [11–14] and relaxed boundary conditions [15] were taken into account.

As a supplement to our previous works [15], here we use the verified code [8, 15], which was a modified approach to [10] via using the complex-matrix-preconditioning technique (also a modified approach to [16] and [17]) to report some *interesting* spectra, which might inspire further research for those cited in [1–3, 5–7, 11–14]

During the period in which we verified our results, we incidentally found certain 'strange' spectra which had never been mentioned in the literature [18]. These spectra, being one pair of eigenvalues for plane Poiseuille flow corresponding to the specific Reynolds number and wavenumber, have almost the same real parts (the phase speed) but different imaginary parts (the amplification factor). Since then, we have begun to search all these spectra in the direction of decreasing Reynolds number and increasing wavenumber, but the behaviour of these strange spectra disappears as the Reynolds number approaches 400. In this paper we only present several specific spectra up to Reynolds number = 500, wavenumber = 1.820. These results could, at least, serve as clues to the study of the normal fluids when we consider the two-fluid system of helium II [15]. Note that, as pointed out by Reddy and Henningson (1993) [19], due to the non-normality of the OS governing operator, even though our approach is linear stability analysis, the results still give the fact that a subcritical transition can occur for the plane Poiseuille flow [5–7, 11–14].

We use the orthogonal polynomial expansion to approximate the governing equations and boundary conditions and solve the eigenvalue problem by using the verified code [8], which used the spectral method [9] based on the Chebyshev-polynomial-expansion approach, since the equation and boundary conditions were discretized. The algebraic equation is

$$
\begin{aligned}
&\frac{1}{24} \sum_{\substack{p=n+4 \\ p\equiv n (\mathrm{mod}\, 2)}}^{N} [p^3(p^2-4)^2 - 3n^2p^5 + 3n^4p^5 + 3n^4p^3 - pn^2(n^2-4)^2]a_p \\
&\qquad - \sum_{\substack{p=n+2 \\ p\equiv n (\mathrm{mod}\, 2)}}^{N} \{[2\alpha^2 + \tfrac{1}{4}\mathrm{i}\alpha R(4f - 4\lambda - c_n - c_{n-1})]p(p^2-n^2) \\
&\qquad - \tfrac{1}{4}\mathrm{i}\alpha R c_n p[p^2 - (n+2)^2] - \tfrac{1}{4}\mathrm{i}\alpha R d_{n-2} p[p^2 - (n-2)^2]\}a_p \\
&\qquad + \mathrm{i}\alpha R n(n-1)a_n + \{\alpha^4 + \mathrm{i}\alpha R[(f-\lambda)\alpha^2 - 2]\}c_n a_n \\
&\qquad - \tfrac{1}{4}\mathrm{i}\alpha^3 R[c_{n-2}a_{n-2} + c_n(c_n + c_{n-1})a_n + c_n a_{n+2}] = 0
\end{aligned}
\qquad (2)
$$

Table 1. Strange spectra for plane Poiseuille flow.

R	α	Mode no	c_r	c_i	$R\alpha$
500	1.820 142 24	17	0.490 069 073	−0.060 422	910.071
		19	0.490 069 076	−0.350 344	
750	1.525 106 28	15	0.435 777 708	−0.038 060	1143.80
		17	0.435 777 709	−0.325 452	
1000	1.368 624 41	15	0.400 199 366	−0.028 452	1368.624
		17	0.400 199 366	−0.307 121	
1200	1.284 273 8	14	0.379 044 858	−0.023 95	1541.129
		16	0.379 044 872	−0.295 44	
1500	1.193 151 27	13	0.354 607 584	−0.019 635	1789.727
		15	0.354 607 585	−0.281 23	
2800	0.990 289 26	10	0.294 247 244	−0.012 0984	2772.81
		12	0.294 247 244	−0.242 8658	
5750	0.820 038 05	9	0.237 490 383	−0.007 8836	4715.219
		10	0.237 490 383	−0.202 685	

for $n \geqslant 0$, $f = 1$, where $c_n = 0$ if $n > 0$, and $d_n = 0$ if $n < 0$, $d_n = 1$ if $n \geqslant 0$. Here, $\lambda \equiv c$ is the complex eigenvalue. The boundary conditions become

$$\sum_{\substack{n=0\\ n\equiv 0 (\mathrm{mod}\, 2)}}^{N} a_n = 0 \qquad \sum_{\substack{n=0\\ n\equiv 0 (\mathrm{mod}\, 2)}}^{N} n^2 a_n = 0 \tag{3}$$

$$\sum_{\substack{n=1\\ n\equiv 1 (\mathrm{mod}\, 2)}}^{N} a_n = 0 \qquad \sum_{\substack{n=1\\ n\equiv 1 (\mathrm{mod}\, 2)}}^{N} n^2 a_n = 0. \tag{4}$$

After we obtained the algebraic system equations ($\boldsymbol{AX} = \boldsymbol{cBX}$), where $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{X}$ are all complex matrices, we modified the Osborne preconditioning algorithm [17], which is for a real matrix, to handle our complex matrices [20].

In brief [8, 15, 18], this algorithm produces a sequence of matrices $\boldsymbol{A}_k$, ($k = 1, 2, \ldots$) diagonally similar to $\boldsymbol{A}$ such that for an irreducible $\boldsymbol{A}$:

(i) $\boldsymbol{A}_f = \lim_{k\to\infty} \boldsymbol{A}_k$ exists and is diagonally similar to $\boldsymbol{A}$,
(ii) $\|\boldsymbol{A}_f\|_2 = \inf(\|\boldsymbol{D}^{-1}\boldsymbol{AD}\|_2)$, where $\boldsymbol{D}$ ranges over the class of all non-singular diagonal matrices,
(iii) $\boldsymbol{A}_f$ is preconditioned in $\|\cdot\|_2$ and
(iv) $\boldsymbol{A}$ and $\boldsymbol{D}^{-1}\boldsymbol{AD}$ produce the same $\boldsymbol{A}_f$.

Then we transform these matrices to Hessenberg form [21] and use the complex QR/LR solver [22, 23] to find the complex eigenvalues related to different Reynolds numbers and wavenumbers. The preliminary verified results of this numerical code [8, 15] had been done in comparison with the bench-mark results of Orszag obtained in 1971. For example, for $R = 10\,000.0$, $\alpha = 1.0$ of the test case, plane Poiseuille flow [10], we obtained the same spectrum as 0.237 526 48 + i0.003 739 67 for $c_r + \mathrm{i}c_i$ [8], which Orszag obtained from CDC 7600 in 1971 [10]. This code did not have the numerical problems [24] which are common in using the spectral method. Then we obtained (through tremendous searching using double-precision machine accuracy) the spectra shown in table 1.

We can say that this kind of strange spectrum will premature any instability mechanism considering the temporal growth of the disturbances [3, 6, 7, 12–14], even though our spectra are for stationary states. This can be easily understood if we go back to the theory of a system of differential equations. Because of this 'degeneracy', the solutions must contain terms of eigensolutions (exponential functions) multiplied by t, and thus favour larger transient growth for this mode. Moreover, this 'degeneracy' might induce something like resonances between vertical velocity perturbation-waves and a vorticity perturbation-wave or other three-dimensional waves [6, 12, 25–27]. All these effects can result in earlier linear flow instability and a further stage (via complicated interactions): transition as demonstrated in [1–3, 5–7, 11–14]. We notice that some boundary conditions might also lead to smaller critical Reynolds number as reported in [15]; our further study will be (i) whether similar effects are observed in the 'strange' spectra and (ii) whether there is any link between the present results and those reported in [28].

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